ECE317 : Feedback and Control

Lecture
System Modelling: dc-dc converter

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Course roadmap

Modeling

- Laplace transform
- Transfer function
- Block Diagram
- Linearization
- Models for systems
  - electrical
  - mechanical
  - example system

Analysis

Stability
- Pole locations
- Routh-Hurwitz

Time response
- Transient
- Steady state (error)

Frequency response
- Bode plot

Design

Design specs

Frequency domain

Bode plot

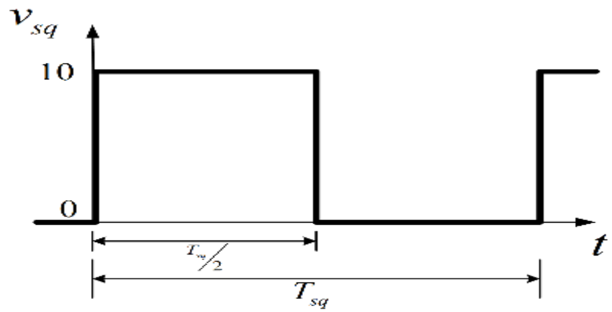
Compensation

Design examples

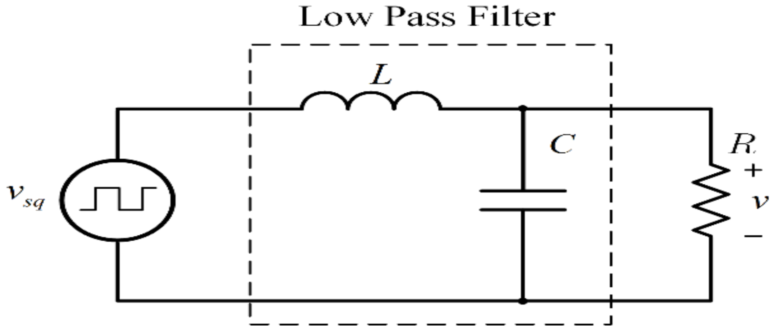
Matlab & PECS simulations & laboratories

Filtering a Square Wave

- Given a square wave input with frequency f_{sq} :



- into a low pass filter with corner frequency f_c :



- If $f_C <<< f_{sq}$ what is v (the voltage across R)?

Consider the spectrum

Fourier Series:

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\left(\frac{2\pi}{T_0}\right)kt}\,dt \quad \textbf{Analysis}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \quad \textbf{Synthesis}$$

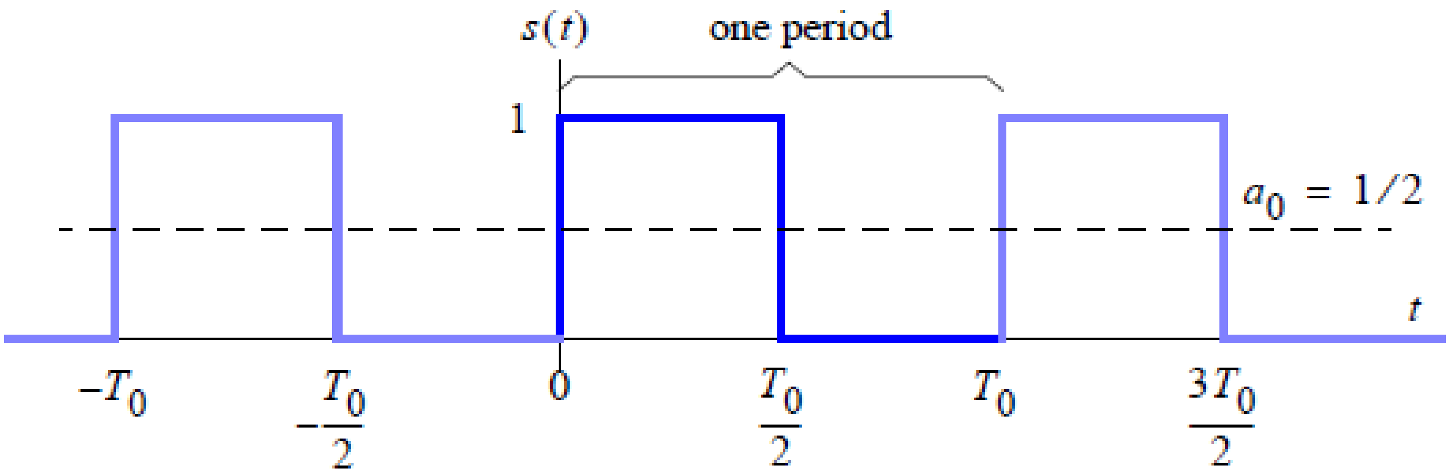
Spectrum cont'd.

The Square Wave

- Here we consider a signal which over one period is given by

$$s(t) = \begin{cases} 1, & 0 \le t < T_0/2 \\ 0, & T_0/2 \le t < T_0 \end{cases}$$

  - This is actually called a 50% duty cycle square wave, since it is *on* for half of its period



Spectrum cont'd.

- We solve for the Fourier coefficients via integration (the Fourier integral)

$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j(2\pi/T_0)kt}dt + 0$$

$$= \frac{1}{T_0}\left[\frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k}\right]\Bigg|_0^{T_0/2} = \frac{1-e^{-j\pi k}}{j2\pi k}$$

- Notice that $e^{-j\pi} = -1$, so

$$a_k = \frac{1-(-1)^k}{j2\pi k} \text{ for } k \neq 0$$

and for $k = 0$ we have

$$a_0 = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j0}dt = \frac{1}{2} \text{ (DC value)}$$

  - This is the average value of the waveform, which is dependent upon the 50% aspect (i.e., halfway between 0 and 1)

Spectrum cont'd.

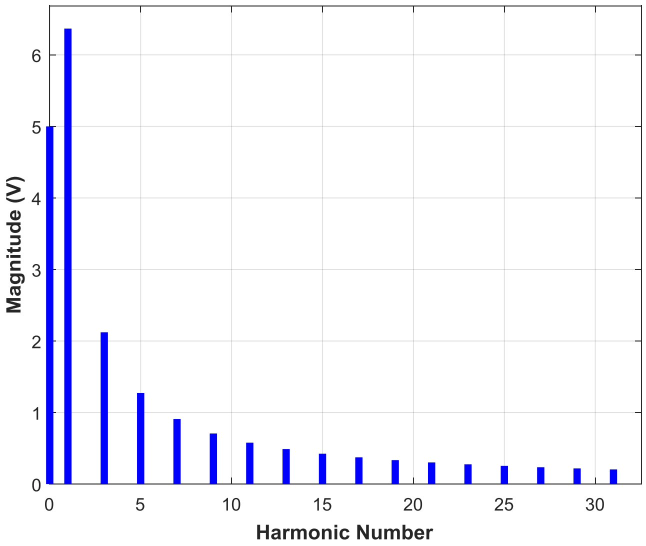
- In summary,

$$a_k = \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1}{j\pi k}, & k = \pm 1, \pm 3, \pm 5, \ldots \\ 0, & k = \pm 2, \pm 4, \pm 6, \ldots \end{cases}$$

- DC (zero freq.) component is ½ the peak
- All even harmonics are 0
- Therefore only odd harmonics are present

Spectrum cont'd.

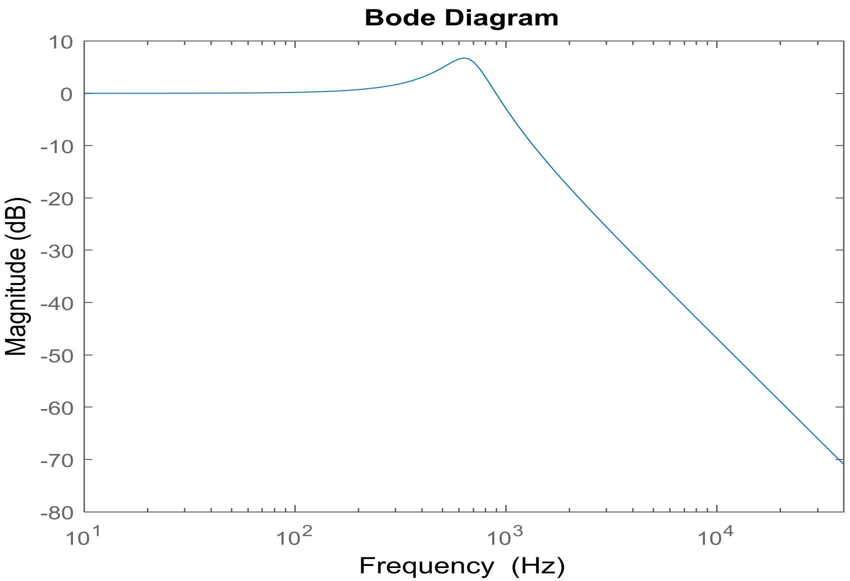
- Amplitude spectrum (for 10 V pk-pk square wave):



- Zero freq. amplitude: $5\ \mathrm{V}$
- 1st harmonic peak amplitude: $2 \times \frac{1}{\pi} = 6.366\ \mathrm{V}$
- 2nd harmonic peak amplitude: $0\ \mathrm{V}$
- 3rd harmonic peak amplitude: $2 \times \frac{1}{3\pi} = 2.122\ \mathrm{V}$

Filtering the square wave

- Low pass filter magnitude response:
- Filter component values: $L = 560\ \mu H, C = 100\ \mu F, R = 5\ \Omega$
- Filter corner frequency: $f_C = 673\ Hz$
- Square wave: $f_{sq} := 40\ kHz$



- Attenuation at 1st harmonic (40 kHz) is -71 dB
- Attenuation at 3rd harmonic (120 kHz) is -90 dB

Filtering the square wave, cont'd

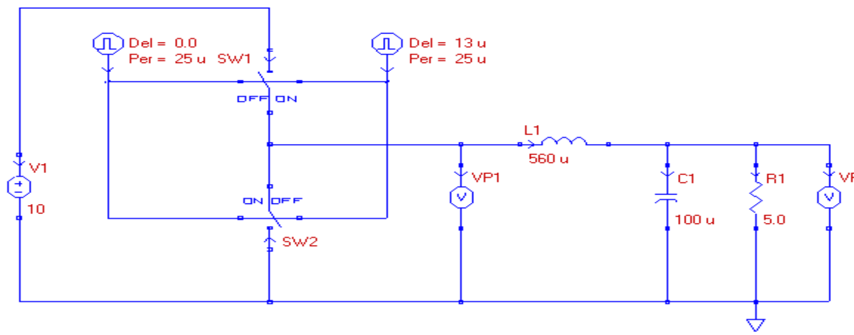
- First harmonic filtered peak amplitude:
- -71 dB attenuation → attenuation by a factor:

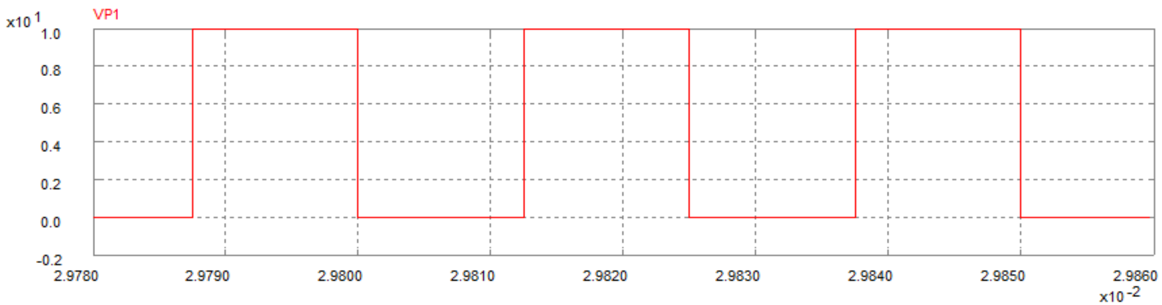
$$10^{\left(-\frac{71}{20}\right)} = 0.000283 \rightarrow$$

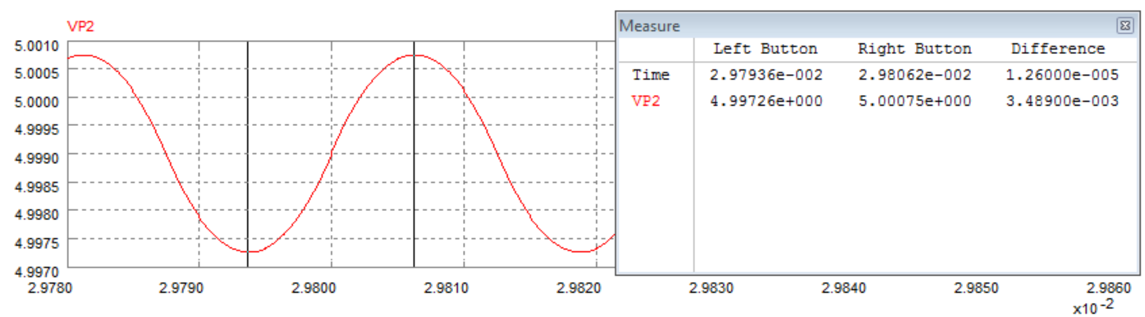
$$6.366 \times 0.000283 = 1.8 \text{ mV}$$

- → peak-peak amplitude: $2 \times 1.8 = 3.6 \text{ mV}$

PECS simulation:







- Peak-peak output ripple: 3.5 mV

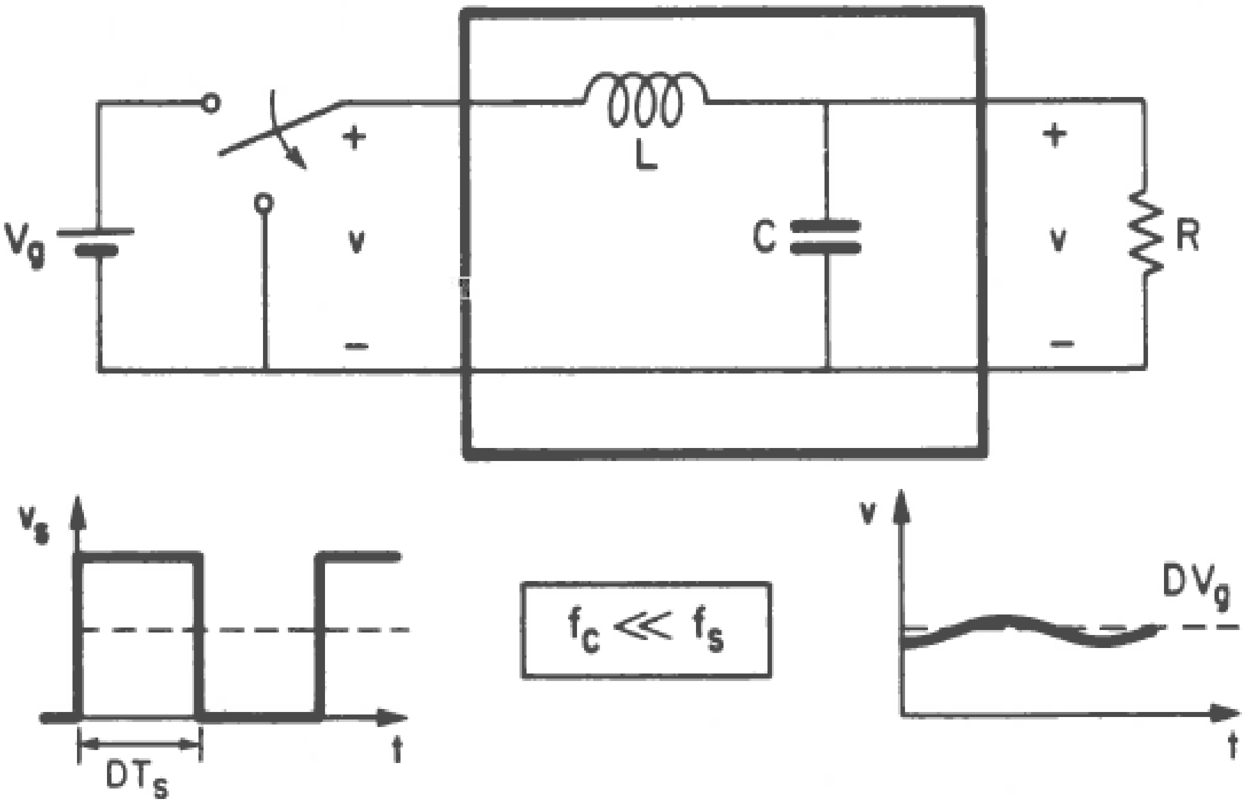
Filtering the square wave, cont'd

- PECS simulation result:
  Peak-peak output ripple: 3.5 mV

- Fourier analysis result:
  First harmonic only filtered pk-pk amplitude: 3.6 mV

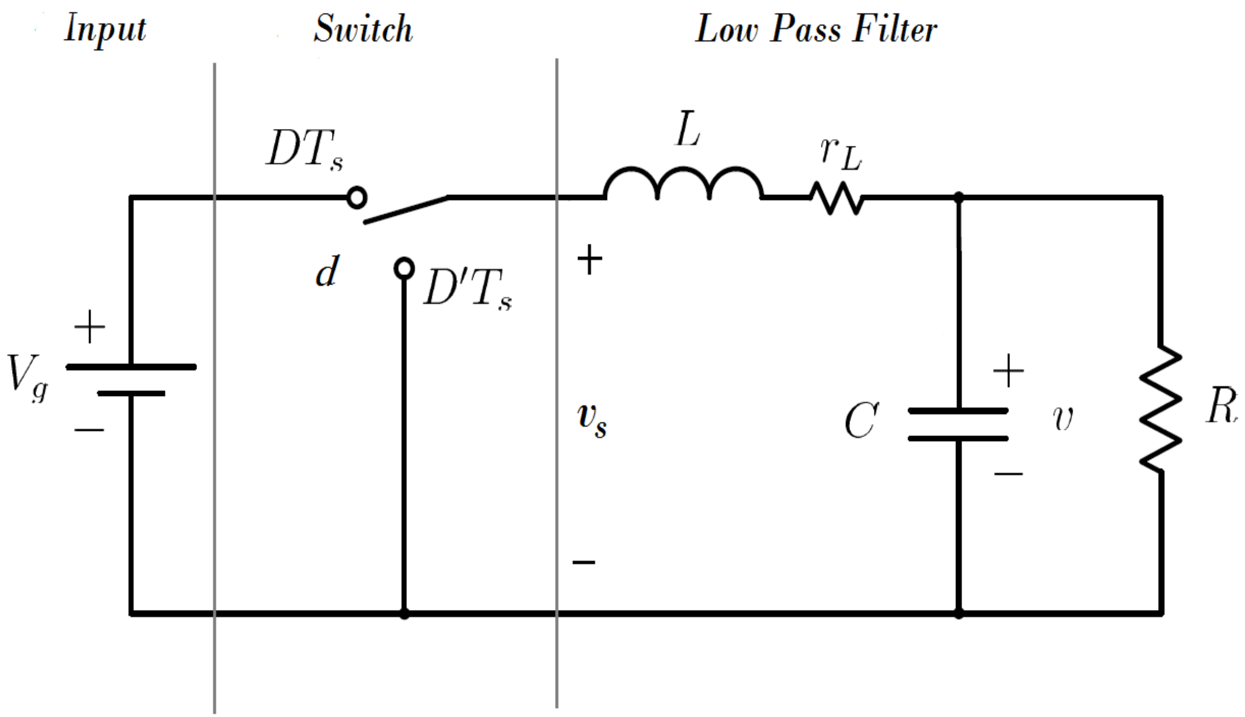
Dc-to-dc Buck Converter

- Convert a DC voltage to a lower DC voltage level at high efficiency
- Chop up the input voltage (Vg) and then filter it
- Inclusion of a single-pole, double-throw switch produces a rectangular waveform to the output filter

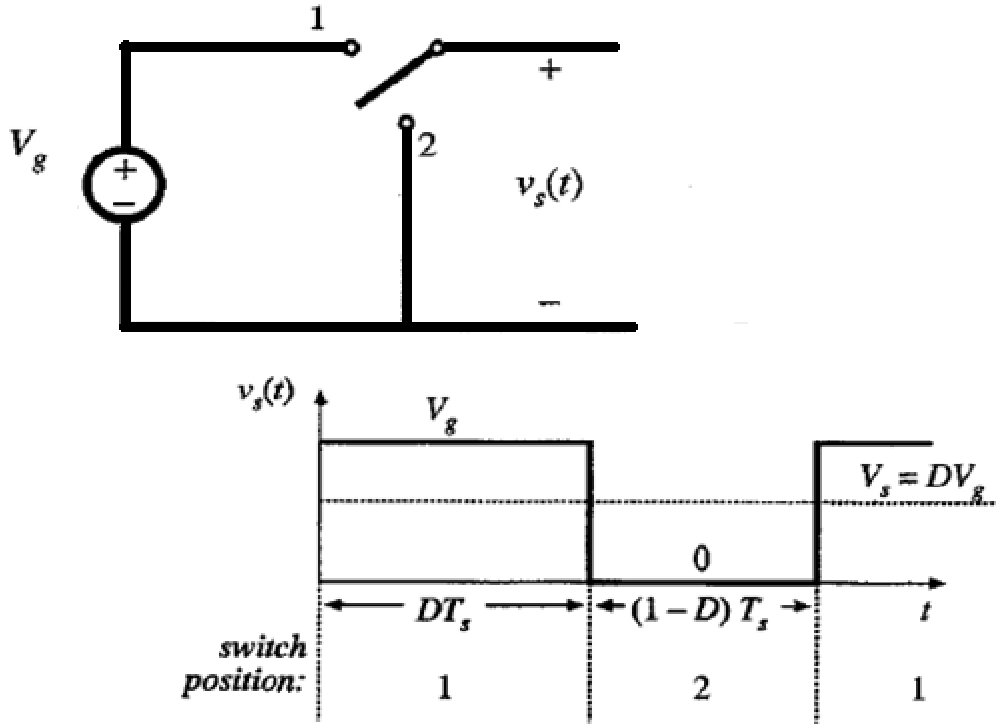


Dc-to-dc Buck Converter, cont'd

- Three sections
  1. Input voltage, vg
  2. Single-pole, double-throw switch
  3. Low pass filter



Average Switch Modelling, cont'd



$$\langle v_s \rangle = \frac{1}{T_s}\int_0^{T_s} v_s\,(t)dt = \langle d \rangle \cdot \langle v_g \rangle$$

$\langle x \rangle$ ← Denotes the average of the variable x.

Note: d and vg are permitted to vary.

Average Switch Modelling, cont'd

Notation:

Let x be a variable of interest, it may represent quantities such as duty ratio, d , input voltage, v_g , input voltage to the low pass filter, v_s , and output current, i_o.

Due to the switching action the system is time varying. To simplify the analysis we will look at the average behavior of the system and produce a time invariant model of the system.

Therefore we are interested in looking at average signals denoted by $\langle \mathrm{x} \rangle$.
We have just derived the relationship: $\langle v_s \rangle = \langle d \rangle \cdot \langle v_g \rangle$

However, to simplify the notation we will drop the angular bracket notation with the understanding that henceforth x now represents an averaged variable. With this simplified notation, the above equation becomes:

$$v_s = d \cdot v_g$$

Average Switch Modelling, cont'd

$$v_s = d \cdot v_g$$

Note that this formula represents the product of two variables. Consequently it is nonlinear. The above is a nonlinear model.

To derive a linear model we will need to linearize around a steady state operating point under the assumption of small-signal deviations from this operating point.

We will use a caret '^' to indicate small deviations so that $\hat{x}$ represents a small deviation away from the steady state operating point X, so that: $\hat{x} = x - X$ or

$$x = X + \hat{x}$$

This states that an averaged variable x in general consists of a steady state average X plus a small signal deviation $\hat{x}$. This relationship will be used next to produce a linear model.

Average Switch Modelling, cont'd

$$v_s = d \cdot v_g$$

Let:

$v_s = V_s + \hat{v}_s$ where $\hat{v}_s << V_s$

$d = D + \hat{d}$ where $\hat{d} << D$

$v_g = V_g + \hat{v}_g$ where $\hat{v}_g << V_g$

Small signal assumption

$$\Rightarrow \quad V_s + \hat{v}_s = (D + \hat{d}) \cdot (V_g + \hat{v}_g)$$

$$= DV_g + D\hat{v}_g + V_g\hat{d} + \hat{d}\hat{v}_g$$

DC term (zero order term): DV_g

First order terms: $D\hat{v}_g + V_g\hat{d}$

Nonlinear term (higher order terms): $\hat{d}\hat{v}_g$

- The product of the two small-signal variations will be small and so will be neglected. This leaves only the linear terms. In this way we have linearized the model.

Average Switch Modelling, cont'd

Equating zero order terms on both sides results in the DC model:

$$V_s = DV_g$$

Since the low pass filter has a DC gain of unity (neglecting losses) the DC converter output voltage V is given by:

$$V = V_s$$

$$\Rightarrow$$

$$V = DV_g$$

Average Switch Modelling, cont'd

Equating first order terms on both sides results in the small-signal model:

$$\hat{v}_s = D\hat{v}_g + V_g\hat{d}$$

$$\Rightarrow$$

$$\frac{\hat{v}_s}{\hat{v}_g} = D, \qquad (\hat{d} = 0)$$

$$and$$

$$\frac{\hat{v}_s}{\hat{d}} = V_g, \qquad \left(\hat{v}_g = 0\right)$$

System Transfer Functions

1) Transfer Function $G_{vg}(s)$:

$$G_{vg}(s) \triangleq \frac{\hat{v}}{\hat{v}_g}$$

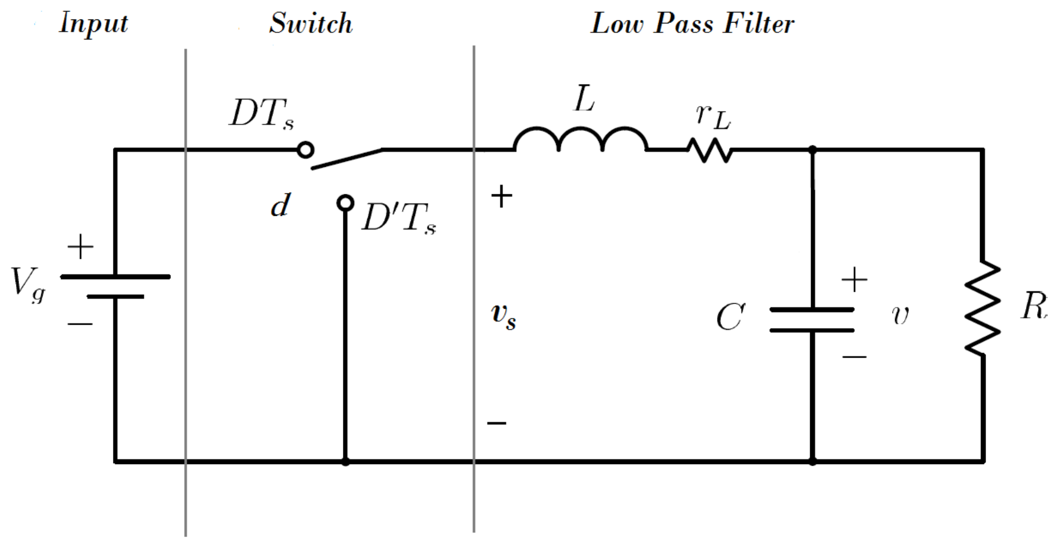
- This transfer function quantifies how variations in the input voltage $\hat{v}_g$ propagate to appear as variations in the output voltage $\hat{v}$.
- input $\hat{v}_g$ represents a disturbance signal

2) Transfer Function $G_{vd}(s)$:

$$G_{vd}(s) \triangleq \frac{\hat{v}}{\hat{d}}$$

- This transfer function quantifies how variations in the duty ratio $\hat{d}$ propagate to appear as variations in the output voltage $\hat{v}$.
- duty ratio $\hat{d}$ represents the control signal

System Transfer Functions

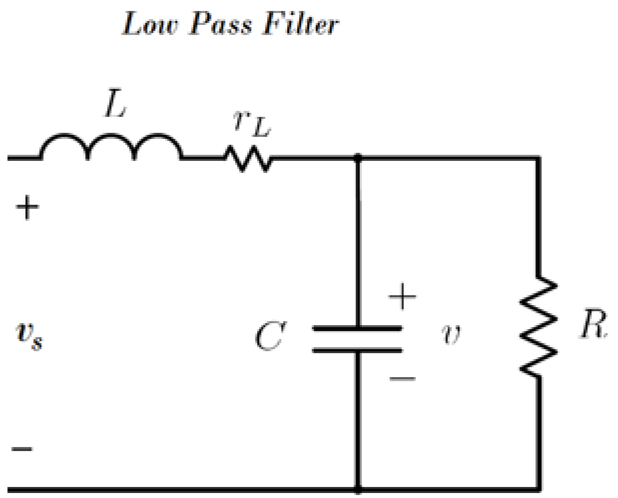


$$G_{vg}(s) \triangleq \frac{\hat{v}}{\hat{v}_g} = \frac{\hat{v}_s}{\hat{v}_g} \cdot \frac{\hat{v}}{\hat{v}_s} = D \cdot G_{LPF}(s)$$

$$G_{vd}(s) \triangleq \frac{\hat{v}}{\hat{d}} = \frac{\hat{v}_s}{\hat{d}} \cdot \frac{\hat{v}}{\hat{v}_s} = V_g \cdot G_{LPF}(s)$$

System Transfer Functions, cont'd

Output section of the Buck converter is a low pass filter:



$$G_{LPF}(s) = \frac{\hat{v}}{\hat{v}_s} = \frac{1}{1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2}$$

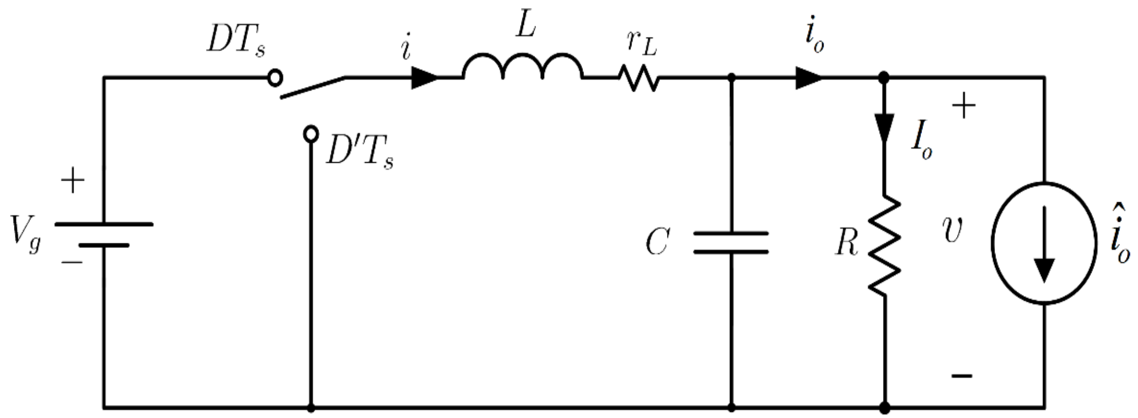
where

$$Q = \frac{\sqrt{LC}}{r_L C + \frac{L}{R}}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

System Transfer Functions, cont'd

3) Transfer Function $\frac{\hat{v}}{\hat{\imath}_o}(s)$:

- The effect of output current variations, $\hat{\imath}_o$, causing output voltage variations, $\hat{v}$, is quantified by transfer function $-Z_{out}$.
- Input $\hat{\imath}_o$ represents a disturbance signal

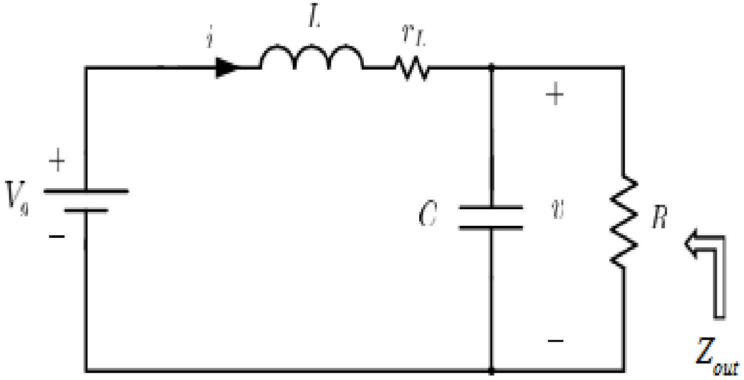


$$Z_{out} = \frac{\hat{v}}{-\hat{\imath}_o}$$

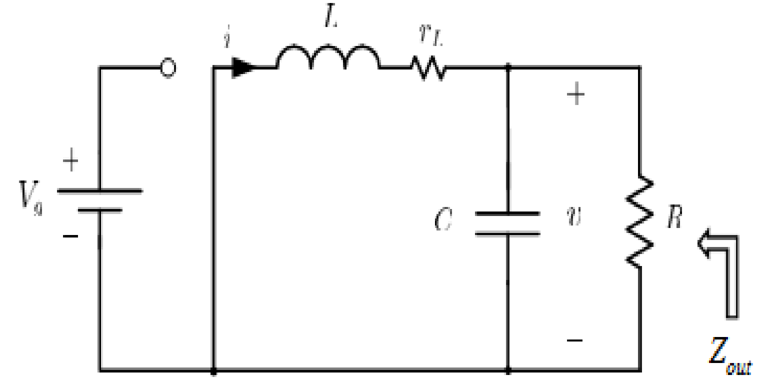
$$\Rightarrow \qquad \frac{\hat{v}}{\hat{\imath}_o}(s) = -Z_{out}$$

System Transfer Functions, cont'd

first subinterval, DT_s:



second subinterval, $D'T_s$:



$$Z_{out} = (sL + r_L) \parallel \frac{1}{sC} \parallel R$$

$\Rightarrow$

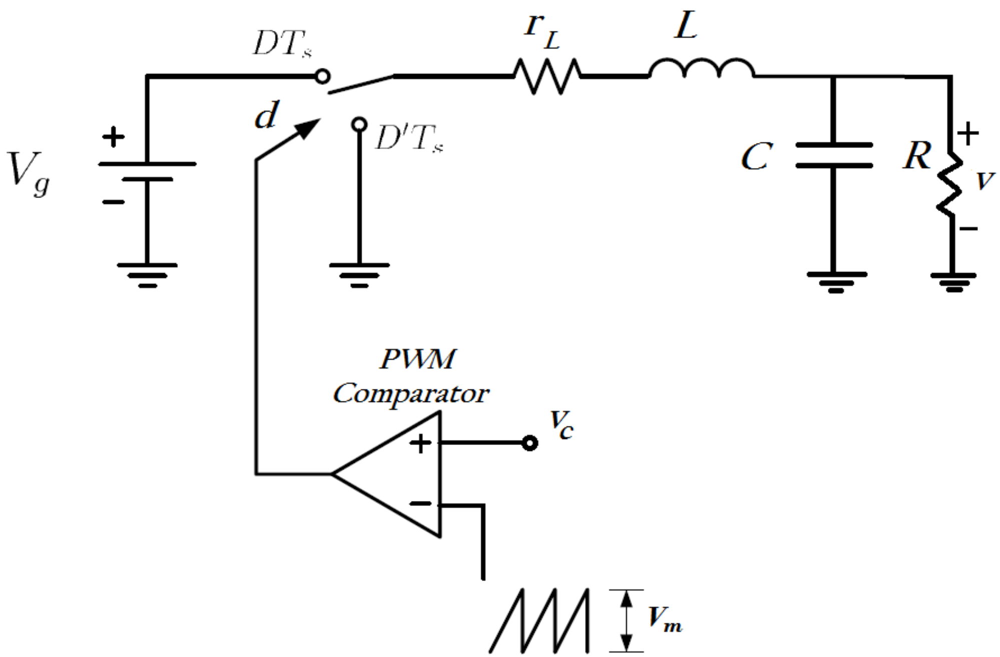
$$-Z_{out} = -\frac{r_L\left(1+\frac{sL}{r_L}\right)}{1+\frac{s}{\omega_0 Q}+\left(\frac{s}{\omega_0}\right)^2}$$

where

$$Q = \frac{\sqrt{LC}}{r_L C+\frac{L}{R}}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

System Transfer Functions, cont'd

Addition of the Pulse Width Modulator:

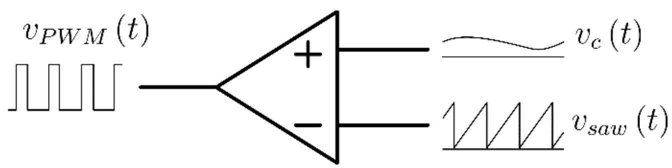


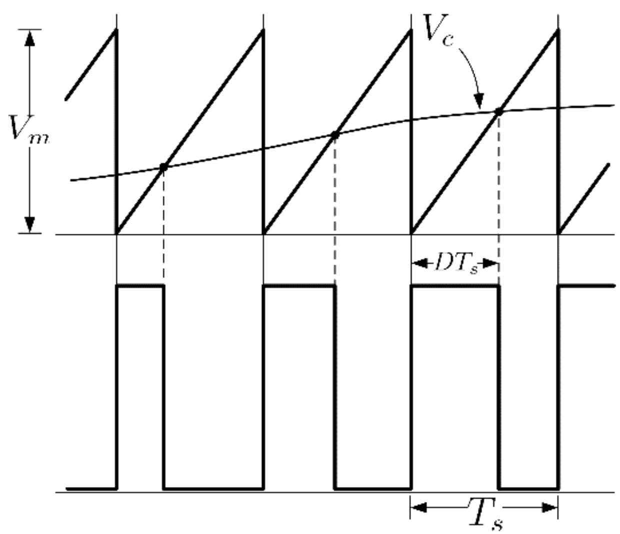
Control-to-output transfer function: $\frac{\hat{v}}{\hat{v}_c} = G_{vd} \cdot G_{PWM} = \frac{\hat{v}}{\hat{d}} \cdot \frac{\hat{d}}{\hat{v}_c}$

System Transfer Functions, cont'd

Pulse Width Modulator (PWM) 'describing function': $G_{PWM} \triangleq \frac{\hat{d}}{\hat{v}_c}$

The modulator consists of a comparator driven by a sawtooth waveform, v_{saw}, at one input and the control signal, v_c, at the other. This produces a rectangular waveform at the output with a duty ratio, d.





Sawtooth waveform has a peak-to-peak amplitude of V_M.

System Transfer Functions, cont'd

- Through a Fourier analysis of the waveform produced by the PWM the 'describing function' can be derived.
- A describing function gives the frequency response of the fundamental component in the output spectrum.
- The result is given by:

$$G_{PWM} \triangleq \frac{\hat{d}}{\hat{v}_c} = \frac{1}{V_M}$$

V_M is the peak-to-peak amplitude of sawtooth waveform.

System Transfer Functions, cont'd

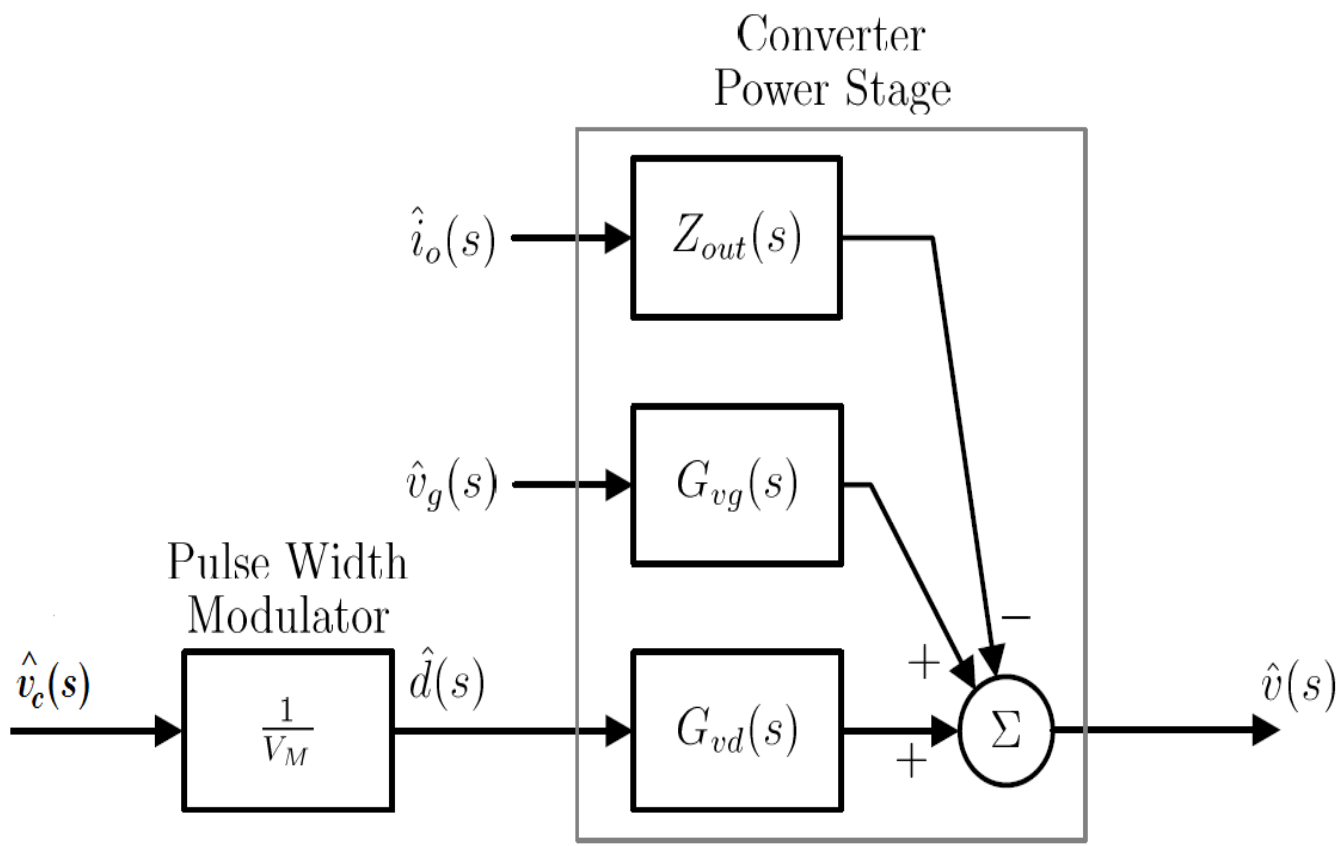
Summary of Transfer Functions of the buck converter and modulator

$G_{vd} \triangleq \frac{\hat{v}}{\hat{d}}$	$\frac{V_g}{\Delta(\mathrm{s})}$
$G_{vg} \triangleq \frac{\hat{v}}{\hat{v}_g}$	$\frac{D}{\Delta(\mathrm{s})}$
$-Z_{out} \triangleq \frac{\hat{v}}{\hat{i}_o}$	$-\frac{r_L\left(1+\frac{sL}{r_L}\right)}{\Delta(\mathrm{s})}$
$G_{PWM} \triangleq \frac{\hat{d}}{\hat{v}_c}$	$\frac{1}{\mathrm{V}_M}$

$$\Delta(\mathrm{s}) = 1 + \frac{s}{\omega_0 Q} + \left(\frac{s}{\omega_0}\right)^2, \quad Q = \frac{\sqrt{LC}}{r_L C + \frac{L}{R}}, \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

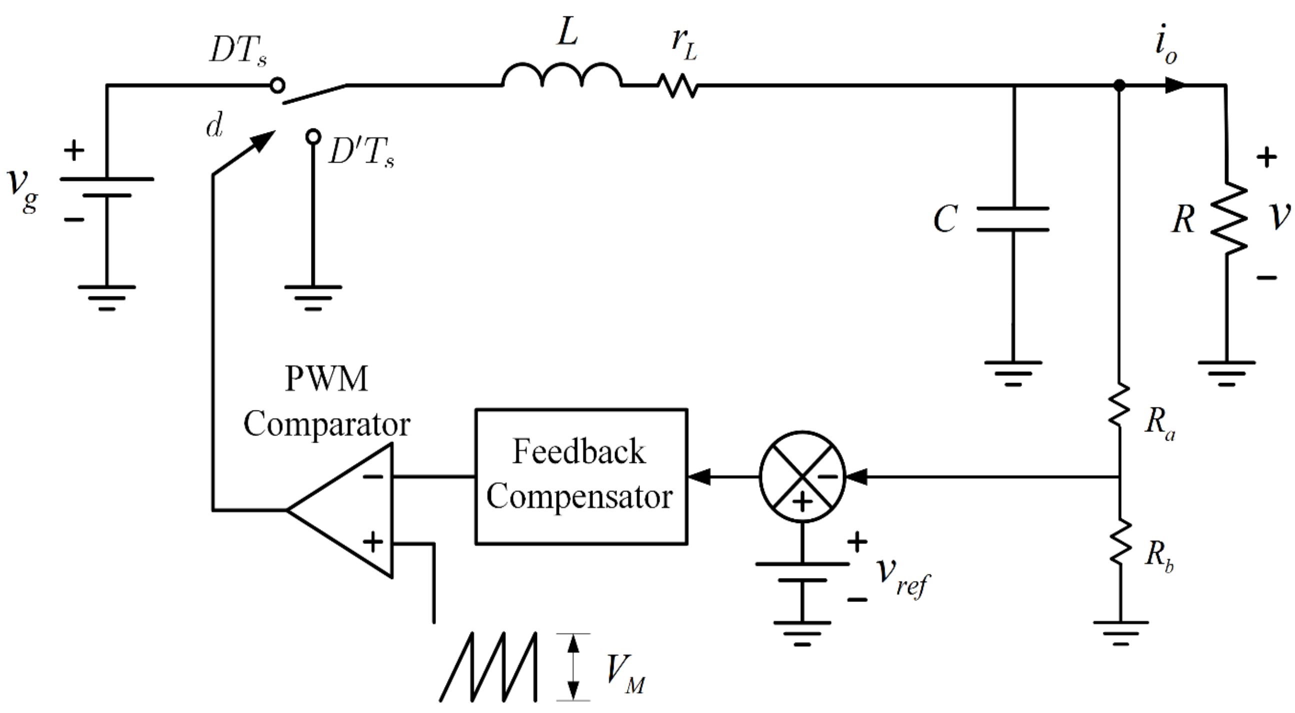
System Transfer Functions, cont'd

Open Loop transfer functions:

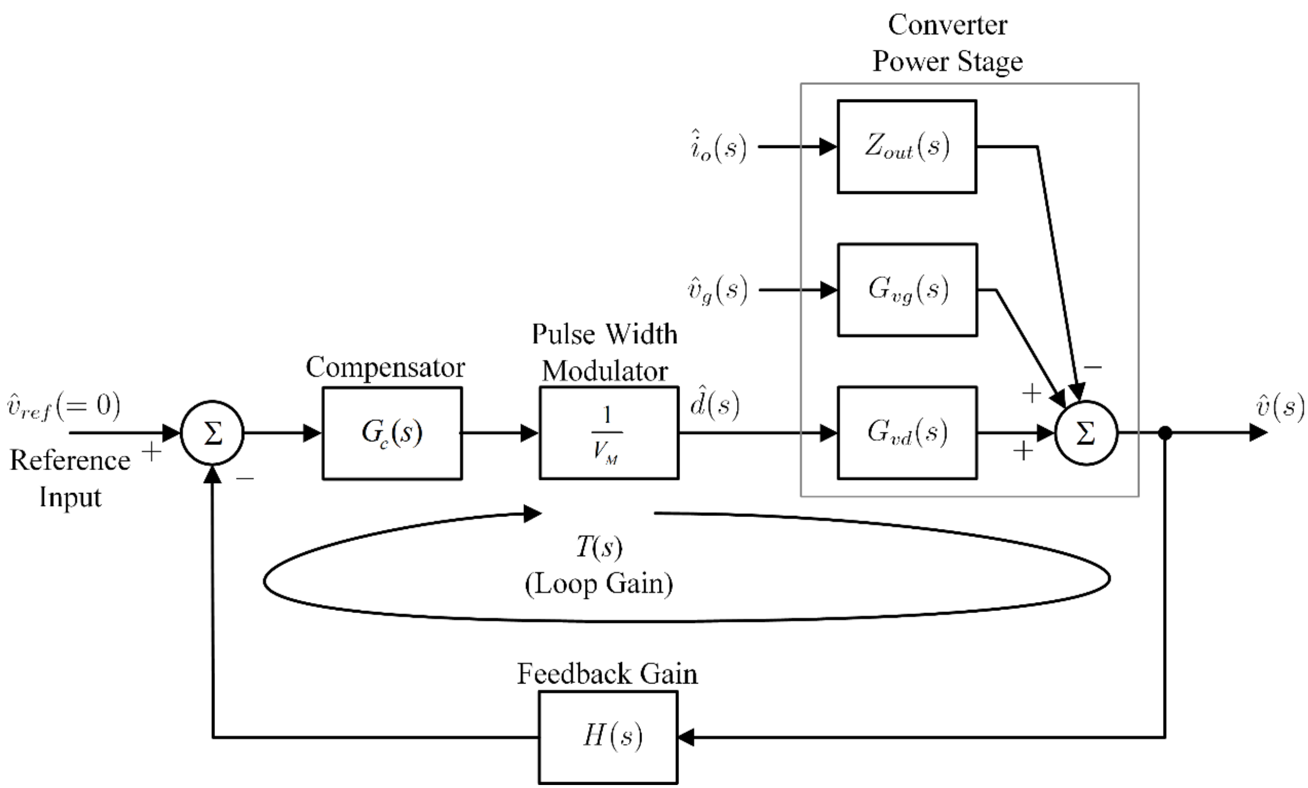


Closed loop system

- The open loop system is closed in a loop through a voltage divider and feedback compensator



Closed loop system: block diagram representation



$H(s)$ **– feedback gain**	$\frac{R_b}{R_a + R_b}$

Summary

- Buck dc-to-dc voltage converter
  - Basic operation of chopping up input voltage and then filtering
  - Used Fourier analysis to determine unfiltered residuals
  - The system has one output, the output voltage, v
  - Identified inputs to the system:

    $\hat{d}$ – duty ratio, the control input

    $\hat{v}_g$ – input voltage, a disturbance input

    $\hat{\imath}_o$ - output current, a disturbance input
  - Determined three transfer functions for the converter:

    $G_{vd}(s)$, $G_{vg}(s)$ and $Z_{out}(s)$.
  - Discussed the 'describing function' analysis to model the PWM
  - Through a process of averaging and linearization a linear time invariant (small-signal) model of the system was obtained

- Next, stability analysis